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An Analysis Of The Development And Application Of Orthogonal Polynomials With An Emphasis On The Legendre Polynomials

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I. Introduction:

The purpose of this paper is to first and foremost explain the concept of orthogonal polynomials to the student who has no significant background in numerical analysis or theoretical physics. Commencing with the common problem of finding a polynomial approximation to a given function on a closed interval, I will endeavor to show a construction of concepts and theorems from calculus, linear algebra, and real analysis which develops the importance of orthogonal polynomials. After proving a popular method for constructing a family of orthogonal polynomials, we will then use the method to derive one of the most basic families, the Legendre polynomials. We will then take a look at the first few Legendre polynomials and examine recurrence relations for generating new ones.

This family has many interesting applications in different fields of mathematics and physics. The most common of these is the role they play in the least square approximation problem on the interval (-1,1). In fact, the set of Legendre polynomials up to degree *n* ultimately provides the best least square approximation to a function f(x) on (-1,1).[1]

The influence of Legendre polynomials also extends to Gaussian quadrature nodes, differential equations, and spherical harmonics, among other topics. And when expanding our outlook to all families of orthogonal polynomials, the topics are almost endless. Consequently, the field of orthogonal polynomials is useful in a variety of ways, and quite valuable to mathematicians, statisticians, and physicists alike.

[1] Hildebrand 274

II. Least Squares Approximation Problem and Linear Independence

When we speak of least squares approximation, we want to find, for a given a function *f* in C[a,b], a polynomial of degree at most *n* that will minimize the expression $\int [f(x) - P_n(x)]^2\,dx$ across the interval [a,b].

See Section (1)

As you can see, for each P_n, this method requires us to solve an (n+1) by (n+1) matrix. Unfortunately, solving this matrix for P_n does not lessen the amount of work needed to solve for P_{n+1}.[2] We will now look at the orthogonal polynomials approach to this problem. First we will need a few definitions and theorems regarding linear independence.

See Section (2)

III. Orthogonal Sets of Functions

Here, we will look at some definitions and theorems that show the properties of orthogonal polynomials. The first definition is that of a weight function, which is very important as it is one of the main characteristics that distinguishes between families of orthogonal polynomials. For instance, the Legendre polynomials that we will be dealing with have the simple weight function $w(x) \equiv 1$ on [-1, 1]. However, the Jacobi, Chebyshev, and Gegenbauer polynomials have more complicated weight functions, usually in the form of $(1 - x^2)^k$ for some predetermined k. The Hermite and Laguerre polynomials have exponential weight functions.[3]

See Section (3)

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[2] Burden 451

[3] Beckmann 41

Before we move on to the construction of orthogonal sets of functions, we need to show some basic properties of even and odd functions that will simplify our work later.

Thm 3: If f and g are even functions, then fg is even.

Proof: Suppose f and g are even functions.

Then $fg(-x) = f(-x)g(-x) = f(x)g(x) = fg(x)$. Thus fg is even.

Thm 4: If f and g are odd functions, then fg is even.

Proof: Suppose f and g are odd functions.

Then $fg(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = fg(x)$. Thus fg is even.

Thm 5: If f is an odd function and g is an even function, then fg is odd.

Proof: Suppose f is odd and g is even.

Then $fg(-x) = f(-x)g(-x) = (-f(x))g(x) = -f(x)g(x) = -fg(x)$. Thus fg is odd.

Also, for future reference, we need to discuss the outcome of even and odd integrals over symmetric intervals. Since the Legendre polynomials are defined on the symmetric interval [-1,1], it is evident that this will come in handy later.

See Section (4)

Now, the following theorem, based on the Gram-Schmidt process[4], describes how to construct orthogonal polynomials on a closed interval [a, b] with respect to a given weight function w. After describing the construction process, we will use the principle

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[4] Solow 354

of mathematical induction to show that the set of polynomials generated are indeed orthogonal. After that we will apply certain conditions to the process to generate a set of orthogonal polynomials known as the Legendre polynomials.

See Section (5)

IV. Applications of Legendre Polynomials

Legendre polynomials are used in Gaussian quadrature, or more specifically, the roots of Legendre polynomials are used as nodes in Gaussian quadrature. To explain, methods of quadrature are aimed at finding more efficient and accurate ways to approximate integrals. For instance, any calculus student is familiar with some of the more basic quadrature methods, such as the mid-point method, Simpson's rule, and Trapezoidal rule. All of these yield approximations to $\int f(x)$ over a given interval [a,b].

These methods become a bit more complicated when the idea of adaptive quadrature is introduced. An efficient technique of adaptive quadrature can distinguish the amount of functional variance and adapt the step size to the varying requirements of a problem. For instance, the nodes get closer together as the function variance starts getting more extreme. As such a method, Gaussian quadrature chooses the points for evaluation in an optimal, rather than equally spaced, manner. More specifically, the nodes $x_1, x_2, \ldots x_n$ that are needed to produce an integral approximation formula to give exact results for any polynomial of degree less than $2n$ on the interval (-1,1), are in fact the roots of the *n*th-degree Legendre polynomial.[5]

Beckmann shows that the Legendre polynomials have an important use in differential equations as well. It turns out that as a direct result from the weight and

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[5] Burden 207

boundary conditions of the Legendre polynomials, they are the solution to the differential equation: $(1 - x^2)d^2y/dx^2 - 2x\,dy/dx + n(n + 1)y = 0$.[6]

The Legendre polynomials are primarily met in the solution of partial differential equations in spherical coordinates.[7] They are also encountered in probability theory, where they are associated with the uniform distribution.

While the uses for the Legendre polynomials are well documented, it is important to realize that there are many other families of orthogonal polynomials, each with their own important uses. For instance, the Chebyshev polynomials are used to generate an algorithm for the efficient calculation of hypergeometric probabilities.[8] Jacobi polynomials are used to evaluate the weights belonging to a class of quadrature rules.[9] And in general, various families of orthogonal polynomials can be used for the analysis of a trend.[10]

The importance of orthogonal polynomials is not merely a thing of the past either. The Rayleigh-Ritz method, first proposed in 1985, uses boundary characteristic orthogonal polynomials to more efficiently analyze the vibration of certain structures. More than one hundred papers that used this method have been reported and discussed over the past twelve years.[11]

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[6] Beckmann 46
[7] Beckmann 75
[8] Alvo 1
[9] Smith 128
[10] Berry 139
[11] Chakraverty 1

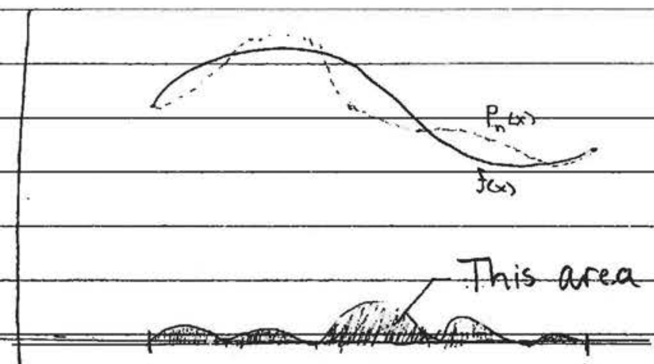
In conclusion, the field of orthogonal polynomials does not appear to be that comprehensive at first. However, the areas of application are so widespread in subject matter and difficulty that one could devote a lifetime to studying them.

[1] Let $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$$= \sum_{k=0}^{n} a_k x^k$$

We want to find coefficients $a_0, a_1, \ldots, a_n$ that will minimize $E = \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k\right)^2 dx$.

Picture:



We can view E as a function of $a_0, a_1, \ldots, a_n$ where to minimize E, we need

$$\frac{\partial E}{\partial a_0} = \frac{\partial E}{\partial a_1} = \cdots = \frac{\partial E}{\partial a_n} = 0.$$ 1

$$E = \int_a^b \left([f(x)]^2 - 2f(x)\sum_{k=0}^{n} a_k x^k + \left[\sum_{k=0}^{n} a_k x^k\right]^2\right) dx$$

$$= \int_a^b [f(x)]^2 dx - 2\sum_{k=0}^{n} a_k \int_a^b f(x)\cdot x^k dx + \int_a^b \left[\sum_{k=0}^{n} a_k x^k\right]^2 dx$$

So for each $j = 0, 1, \ldots, n$

$$\frac{\partial E}{\partial a_j} = -2\int_a^b f(x)\cdot x^j dx + 2\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx$$

1- Solow 205, Def 4.9
2- Burden 451, Thm 8.2

To find $P_n(x)$, the $(n+1)$ linear normal equations $\sum_{k=0}^{n} a_k \int_a^b x^{j+k}\,dx = \int_a^b f(x)\cdot x^j\,dx$, $j=0,1,\dots,n$ must be solved for each of the $(n+1)$ unknowns a_j. (END SECTION [1])

2] Def 1: The set of functions $\{\phi_0, \phi_1, \dots, \phi_n\}$ is linearly independent on $[a,b]$ if whenever $c_0\phi_0(x) + c_1\phi_1(x) + \dots + c_n\phi_n(x) = 0$ for all $x \in [a,b]$, $c_0 = c_1 = \dots = c_n = 0$.[1]

Thm 1: Suppose for each $j = 0, 1, \dots, n$, ϕ_j is a polynomial of degree j. Then $\{\phi_0, \phi_1, \dots, \phi_n\}$ is linearly independent on any interval $[a,b]$.[2]

Proof: Suppose $c_0, c_1, \dots, c_n$ are real numbers for which $P(x) = c_0\phi_0(x) + c_1\phi_1(x) + \dots + c_n\phi_n(x) = 0$ for all $x \in [a,b]$. Since $P(x) \equiv 0$ on $[a,b]$, the coefficient of x^n is zero. (because

1. Solow 214, Thm 4.9a
2. Solow 208, Def 4.10

showing that the set $\{\phi_0, \phi_1, \ldots, \phi_n\}$ spans Π_n. Since the set is linearly independent, and since Π_n has dimension $n+1$, $\{\phi_0, \phi_1, \ldots, \phi_n\}$ forms a basis for Π_n.[1] Thus, by the definition of basis,[2] any polynomial in Π_n can be written as a linear combination of $\phi_0, \phi_1, \ldots, \phi_n$.// (END SECTION [2])

1. Burden 453, Def 8.4
2. Burden 454, Def 8.5

Def 3:

[3] An integrable function w is called a weight function on the interval I if for all $x \in I$, $w(x) \geq 0$, and $w(x)$ cannot be the zero function on any subinterval of I, (including I itself).[1]

Def 4: $\{\phi_0, \phi_1, \dots, \phi_n\}$ is an orthogonal set of functions on $[a, b]$ with respect to the weight function w if:

$$\int_a^b w(x) \cdot \phi_j(x) \cdot \phi_k(x)\, dx = \begin{cases} 0 & \text{whenever } j \neq k \\ \alpha_k > 0 & \text{whenever } j = k \end{cases}$$

Also, if $\alpha_k = 1$ for each $k = 0, 1, \dots, n$ the set is orthonormal.[2] (END SECTION [3])

4] <u>Thm 6</u>: If f is an integrable, odd function defined on the interval $[-a,a]$ for some $a>0$, then $\int_{-a}^{a} f(x)\,dx = 0$.

<u>Proof</u>: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$= \int_{0}^{-a} -f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$= \int_{0}^{-a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$

$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$ ///

<u>Thm 7</u>: If f is an integrable, even function defined on the interval $[-a,a]$ for some $a>0$, then $\int_{-a}^{a} f(x)\cdot dx = 2\int_{0}^{a} f(x)\cdot dx$.

<u>Proof</u>: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\cdot dx + \int_{0}^{a} f(x)\,dx$

$= -\int_{0}^{-a} f(x)\cdot dx + \int_{0}^{a} f(x)\,dx$

$= -\int_{0}^{-a} f(-x)\cdot dx + \int_{0}^{a} f(x)\cdot dx$

$= \int_{0}^{a} f(x)\cdot dx + \int_{0}^{a} f(x)\cdot dx$

$= 2\int_{0}^{a} f(x)\cdot dx$ ///

(END SECTION [4])

1. Burden 454, Thm 8.6

5] Thm 8: The set of polynomials defined in the following way is orthogonal on [a, b] with respect to the weight function ω.[1]

$\phi_0(x) \equiv 1$, $\phi_1(x) = x - B_1$ for each x in [a, b]

where $B_1 = \dfrac{\int_a^b x \cdot \omega(x) [\phi_0(x)]^2 dx}{\int_a^b \omega(x) \cdot [\phi_0(x)]^2 dx}$.

and when $k \geq 2$, $\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k \phi_{k-2}(x)$

for each x in [a, b] where

$B_k = \dfrac{\int_a^b x \cdot \omega(x) \cdot [\phi_{k-1}(x)]^2 dx}{\int_a^b \omega(x) \cdot [\phi_{k-1}(x)]^2 dx}$ and $C_k = \dfrac{\int_a^b x \cdot \omega(x) \phi_{k-1}(x) \phi_{k-2}(x) \cdot dx}{\int_a^b \omega(x) \cdot [\phi_{k-2}(x)]^2 dx}$

Proof: First, we will show that $\{\phi_0, \phi_1\}$ is orthogonal on [a, b] with respect to ω.

Based on the definition of orthogonality, we need to show three things:

(1.) $\int_a^b \omega(x) \cdot [\phi_0(x)]^2 dx = \alpha_0$ for some $\alpha_0 > 0$.

(2) $\int_a^b w(x) \cdot [\phi_1(x)]^2 dx = \alpha_1$ for some $\alpha_1 > 0$.

(3) $\int_a^b w(x) \cdot \phi_0(x) \cdot \phi_1(x) \cdot dx = 0$.

For (1) it follows from the definition of weight function that $\int_a^b w(x) [\phi_0(x)]^2 \cdot dx = \int_a^b w(x) \cdot dx > 0$.

For (2) we must recognize that

$$\phi_1(x) = x - \frac{\int_a^b x \cdot w(x) \cdot dx}{\int_a^b w(x) \cdot dx} \quad \text{for all } x \in [a,b].$$

and thus $\phi_1(x)$ is a linear function.

We have that $\forall x \in [a,b]\ (w(x) \geq 0)$. Also we have that $\forall x \in [a,b]\ ([\phi_1(x)]^2 \geq 0)$. Thus $\forall x \in [a,b]$ $(w(x) \cdot [\phi_1(x)]^2 \geq 0)$. Clearly, $\int_a^b w(x) \cdot [\phi_1(x)]^2 \geq 0$.

Now, there exists some subinterval $(a_1, b_1) \subset [a,b]$ where $w(x) > 0$ for all $x \in (a_1, b_1)$.

And on $[a,b]$, $[\phi_1(x)]^2$ can have a maximum of one zero (because it has degree 2 and never crosses the x-axis). So if $[\phi_1(x)]^2$ has no zeroes on (a_1, b_1), then $\int_{a_1}^{b_1} w(x)[\phi_1(x)]^2 dx > 0 \longrightarrow$

$\int_a^b w(x)\cdot[\phi_1(x)]^2 dx > 0$. If it has one zero on

(a_1, b_1) let us call it x_1. Then $\int_{a_1}^{x_1} w(x)\cdot[\phi_1(x)]^2 dx > 0$

$\longrightarrow \int_a^b w(x)\cdot[\phi_1(x)]^2 dx > 0$.

(3) is fairly straightforward:

$$
\begin{aligned}
\int_a^b w(x)\cdot\phi_0(x)\cdot\phi_1(x)\cdot dx &= \int_a^b w(x)\cdot\left[x - \frac{\int_a^b x\, w(x)\, dx}{\int_a^b w(x)\cdot dx}\right] dx \\
&= \int_a^b \left[w(x)\cdot x - \frac{w(x)\int_a^b x\cdot w(x)\cdot dx}{\int_a^b w(x)\, dx}\right] dx \\
&= \int_a^b w(x)\cdot x\cdot dx - \int_a^b \left[\frac{w(x)\cdot\int_a^b x\cdot w(x)\cdot dx}{\int_a^b w(x)\cdot dx}\right] dx \\
&= \int_a^b w(x)\, x\cdot dx - \frac{\int_a^b x\cdot w(x)\, dx\cdot\cancel{\int_a^b w(x)\, dx}}{\cancel{\int_a^b w(x)\, dx}} \\
&= \int_a^b w(x)\, x\, dx - \int_a^b w(x)\, x\cdot dx \\
&= 0.
\end{aligned}
$$

Now continuing with the process of induction, we need to show that if

$\{\phi_0, \phi_1, \dots, \phi_n\}$ is a set of orthogonal polynomials constructed in this manner, then $\{\phi_0, \phi_1, \dots, \phi_{n+1}\}$ is as well.

Suppose $\{\phi_0, \phi_1, \dots, \phi_n\}$ is a set of orthogonal polynomials on $[a, b]$. We need to show that for each $k = 0, 1, \dots, n$, $\int_a^b w(x) \cdot \phi_k(x) \cdot \phi_{n+1}(x) \cdot dx = 0$.

Now, since the recursion formula tells us that $\phi_{n+1}(x) = (x - B_{n+1})\phi_n(x) - C_{n+1}\phi_{n-1}(x)$, we can sub this value into the integral and expand to get:

$$\underbrace{\left[\int_a^b w(x) \cdot x \cdot \phi_k(x) \cdot \phi_n(x) \cdot dx\right]}_{X} - \underbrace{\left[B_{n+1} \int_a^b w(x) \cdot \phi_k(x) \cdot \phi_n(x)\, dx\right]}_{Y} - \underbrace{\left[C_{n+1} \int_a^b w(x) \cdot \phi_k(x)\, \phi_{n-1}(x)\, dx\right]}_{Z}$$

for each $k = 0, 1, \dots, n$.

Now, think of this equation in the form $X - Y - Z$ as shown above.

Lets look at $X = \int_a^b w(x) \cdot x \cdot \phi_k(x) \cdot \phi_n(x) \cdot dx$ for each $k = 0, 1, \dots, n$. We will show in

1. see Corrolary 8.1

the upcoming corollary[1] that $X = 0$ when $k = 0, 1, \ldots, n-2$. For now, we will make this claim. Now $Y = B_{n+1} \int_a^b w(x)\phi_k(x)\phi_n(x)\,dx$ for each $k = 0, 1, \ldots, n$. Therefore, $Y =$

$$\frac{\int_a^b x \cdot w(x)\,[\phi_n(x)]^2 \cdot dx \cdot \int_a^b w(x)\,\phi_k(x)\cdot\phi_n(x)\,dx}{\int_a^b w(x)\cdot[\phi_n(x)]^2\,dx}$$

$= 0$ when $k = 0, 1, \ldots, n-1$ and

$= \int_a^b x \cdot w(x) \cdot [\phi_n(x)]^2 \cdot dx$ when $k = n$.

$Z = C_{n+1} \int_a^b w(x)\cdot\phi_k(x)\cdot\phi_{n-1}(x)\,dx$ for each $k = 0, 1, \ldots, n$.

$$= \frac{\int_a^b x\cdot w(x)\cdot\phi_n(x)\,\phi_{n-1}(x)\cdot dx \cdot \int_a^b w(x)\cdot\phi_k(x)\cdot\phi_{n-1}(x)\,dx}{\int_a^b w(x)\cdot[\phi_{n-1}(x)]^2\,dx}$$

$= 0$ when $k = 0, 1, \ldots, n-2$, and

$= \int_a^b x\cdot w(x)\cdot\phi_n(x)\cdot\phi_{n-1}(x)\cdot dx$ when $k = n-1$, and

$= 0$ when $k = n$.

Now, returning our attention to the form $X - Y - Z$, we get that whenever

$k = 0, 1, \ldots, n-2$, $X - Y - Z = 0$. When $k = n-1$, $X - Y - Z = \int_a^b w(x) \cdot x \, \phi_{n-1}(x) \cdot \phi_n(x) \cdot dx - \int_a^b x \cdot w(x) \cdot \phi_n(x) \cdot \phi_{n-1}(x) \, dx = 0$. And when $k = n$,

$$X - Y - Z = \int_a^b w(x) \cdot x \cdot [\phi_n(x)]^2 dx - \int_a^b x \cdot w(x) \, [\phi_n(x)]^2 dx = 0.$$

$\therefore$ For each $k = 0, 1, \ldots, n$, $\int_a^b w(x) \cdot \phi_k(x) \phi_{n+1}(x) dx = 0$.

Now finally we must show that

$\int_a^b w(x) [\phi_{n+1}(x)]^2 \cdot dx = \alpha_{n+1}$ for some $\alpha_{n+1} > 0$.

Clearly, the product $w(x) [\phi_{n+1}(x)]^2 \geq 0$ for all $x \in [a, b]$. Now, there exists some subinterval $(a_1, b_1) \subset [a, b]$ where $w(x) > 0$ for all $x \in (a_1, b_1)$. And since $[\phi_{n+1}(x)]^2$ is a polynomial of degree $2(n+1)$, and never crosses the x-axis, it has a maximum of $n+1$ zeroes on $[a, b]$. Suppose it has no zeroes on (a_1, b_1). Then $\int_{a_1}^{b_1} w(x) \cdot [\phi_{n+1}(x)]^2 dx > 0$

which implies $\int_a^b w(x)[\phi_{n+1}(x)]^2 dx > 0$. Suppose it has at least one zero on (a_1, b_1). Lets call the smallest one x_0. Then

$$\int_{a_1}^{x_0} w(x)[\phi_{n+1}(x)]^2 dx > 0 \longrightarrow \int_a^b w(x)\cdot[\phi_{n+1}(x)]^2 dx > 0.$$

$\therefore \{\phi_0, \phi_1, \ldots, \phi_{n+1}\}$ is a set of orthogonal polynomials whenever $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is.

$\therefore$ By PMI, the set $\{\phi_0, \phi_1, \ldots, \phi_n\}$ generated by the method described in Thm 8, is a set of orthogonal polynomials.

Corollary 8.1

$$\int_a^b w(x)\cdot x\cdot \phi_k(x)\cdot \phi_n(x)\,dx = 0 \quad \text{when } k = 0, 1, \dots, n-2$$

Proof Let $k \in \{0, 1, \dots, n-2\}$. By Thm 1, $\{\phi_0, \phi_1, \dots, \phi_{k+1}\}$ is linearly independent on $[a, b]$. Thus $\{\phi_0, \phi_1, \dots, \phi_{k+1}\}$ forms a basis for Π_{k+1}. Thus every polynomial of degree $k+1$ can be written as a linear combination of $\phi_0, \phi_1, \dots, \phi_{k+1}$. Thus, for some constants $c_0, c_1, \dots, c_{k+1}$, $x\cdot\phi_k(x) = \sum_{i=0}^{k+1} c_i \phi_i(x)$. Thus $\int_a^b w(x)\cdot x\cdot \phi_k(x)\phi_n(x)\,dx$

$= \int_a^b w(x)\cdot\phi_n(x)\cdot\left(\sum_{i=0}^{k+1} c_i\phi_i(x)\right)dx$

$= \int_a^b w(x)\cdot\phi_n(x)\cdot c_0\phi_0(x)\cdot dx + \dots\dots + \int_a^b w(x)\phi_n(x)c_{k+1}\phi_{k+1}(x)\cdot dx$

$= c_0\int_a^b w(x)\cdot\phi_0(x)\cdot\phi_n(x)\cdot dx + \dots\dots + c_{k+1}\int_a^b w(x)\phi_n(x)\phi_{k+1}(x)\cdot dx$

$= c_0\cdot(0) + c_1\cdot(0) + \dots\dots + c_{k+1}\cdot(0)$

$= 0$ // From this argument, we see that $\int_a^b w(x)\cdot\phi_n(x)\cdot p_k(x)\cdot dx = 0$ where p_k is any polynomial in Π_{n-1}.

1. Hildebrand 273.

The set of Legendre polynomials, $\{P_n\}$, is orthogonal on $[-1,1]$ with respect to the weight function $w(x) \equiv 1$. The classical definition of the Legendre polynomials requires that $P_n(1) = 1$ for each $n \in I$.[1] By using Thm 8, and disregarding the boundary condition, we will generate a constant mutiple of the Legendre polynomials. For application purposes though, this difference is just a matter of convention. (Later, we will look at the exact Legendre polynomials and the recursive relation used to generate them.) According to Thm 8,

$P_0(x) \equiv 1$ on $[-1,1]$.

$$P_1(x) = X - B_1 = X - \frac{\int_{-1}^{1} x \cdot dx}{\int_{-1}^{1} 1\, dx} = X - 0 = X.$$

$$P_2(x) = (x - B_2)\cdot P_1(x) - C_2 P_0(x)$$

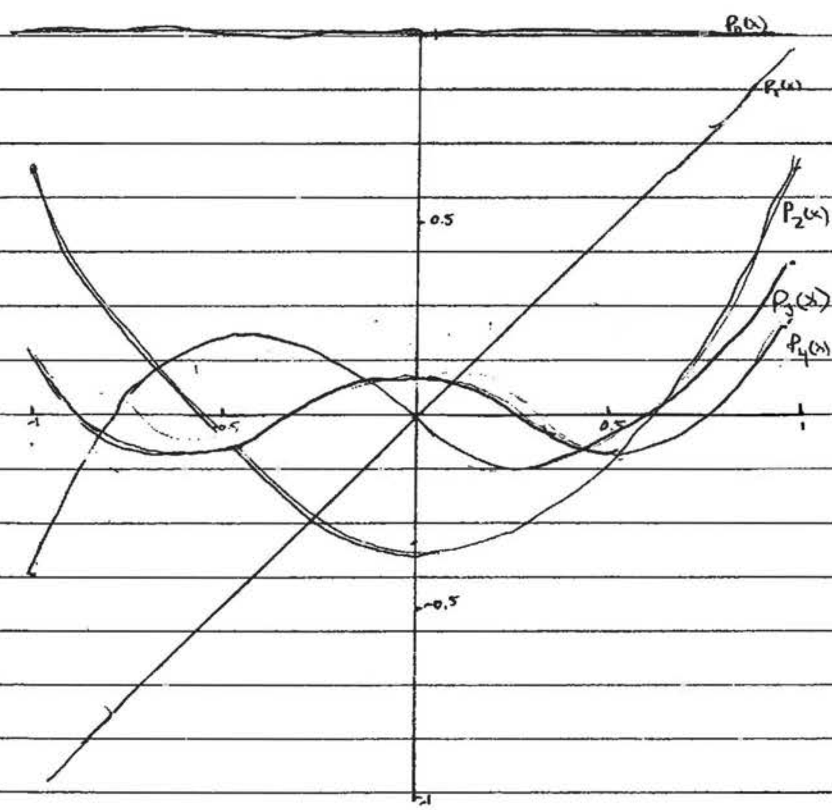
$$= \left(x - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx}\right)\cdot x - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} = x^2 - \frac{1}{3}$$

Continuing in this fashion we find that

$P_3(x) = x^3 - \frac{3}{5}x$, $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$,

and $P_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$ to name a few.

Below is a rough sketch of $P_0(x)$ through $P_4(x)$



Thm 9: (Refinement of Thm 8 for a special case).

The set of polynomials $\{\phi_0, \phi_1, \dots, \phi_n\}$ defined in the following way is orthogonal on $[-a, a]$ (where $a>0$) with respect to the weight function $w(x) \equiv 1$:

$\phi_0(x) \equiv 1$, $\phi_1(x) = x$ for each $x \in [-a, a]$

and when $k \geq 2$,

$\phi_k(x) = x \cdot \phi_{k-1}(x) - C_k \phi_{k-2}(x)$ for each $x \in [-a, a]$

Where $C_k = \dfrac{\int_0^a x \cdot \phi_{k-1}(x) \cdot \phi_{k-2}(x)\, dx}{\int_0^a [\phi_{k-2}(x)]^2 \cdot dx}$.

Proof Applying the interval and weight function to the process described in Thm 8, we get that $B_1 = \dfrac{\int_{-a}^{a} x \cdot dx}{\int_{-a}^{a} 1\, dx} = \dfrac{0}{2a} = 0$.

Thus, $\phi_0(x) \equiv 1$ and $\phi_1(x) = x$ for each $x \in [-a, a]$

Also, when $K \geq 2$, we have

$$B_k = \frac{\int_{-a}^{a} x \cdot [\phi_{k-1}(x)]^2 \cdot dx}{\int_{-a}^{a} [\phi_{k-1}(x)]^2 dx}.$$ But $x \cdot [\phi_{k-1}(x)]^2$ is odd

(by Thm 5) and $\int_{-a}^{a} x [\phi_{k-1}(x)]^2 \cdot dx = 0$ (by Thm 6).

Thus $B_k = 0$ for all $k \geq 2$.

Also, when $k \geq 2$,

$$C_k = \frac{\int_{-a}^{a} x \cdot \phi_{k-1}(x) \cdot \phi_{k-2}(x) \cdot dx}{\int_{-a}^{a} [\phi_{k-2}(x)]^2 dx} = \frac{2\int_{0}^{a} x \cdot \phi_{k-1}(x) \cdot \phi_{k-2}(x) \cdot dx}{2\int_{0}^{a} [\phi_{k-2}(x)]^2 \cdot dx}$$

$$= \frac{\int_{0}^{a} x \cdot \phi_{k-1}(x) \cdot \phi_{k-2}(x) \cdot dx}{\int_{0}^{a} [\phi_{k-2}(x)]^2 dx} \quad ///$$

This method saves us some work when dealing with approximations on symmetric intervals.

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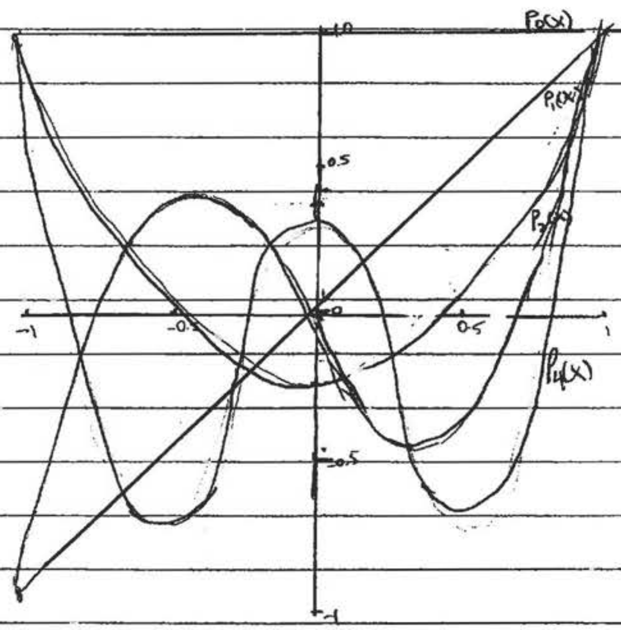
Hildenbrand outlines the origin of the classical definition of Legendre polynomials. Starting with the same least squares approximation problem, he searchs for a polynomial, $\phi_r(x)$ (of degree r), such that $\int_a^b w(x)\cdot\phi_r(x)\cdot q_{r-1}(x)\cdot dx = 0$ where q_{r-1} is an arbitrary polynomial in Π_{r-1}. After integrating by parts r times, and deriving a differential equation related to $\phi_r(x)$, he applies the Legendre conditions (namely the ones we discussed before) to the set of $\phi_r(x)$'s. From the differential equation conditions, it is found that the rth Legendre polynomial is given by $P_r(x) = \frac{1}{2^r r!}\frac{d^r}{dx^r}(x^2-1)^r$. [1]

The constant term $\frac{1}{2^r \cdot r!}$ is chosen to fit the condition that $P_n(1) = 1$ for each $n \in I$. Using this method to generate the Legendre polynomials, we get $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, and $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$ as the first five Legendre polynomials.[1] Here is a rough sketch of P_0 to P_4 on the interval $[-1, 1]$.



1. Hildebrand 273
2. Hildebrand 274

Additional $P_n(x)$'s can be determined from the recurrence formula:

$P_{r+1}(x) = \frac{2r+1}{r+1} P_r(x) - \frac{r}{r+1} P_{r-1}(x)$.[1]

While the $P_n(x)$'s that we derived from Thm 8 give very good least square approximations, Hildebrand concludes that the best possible value of $P_n(x)$ for a given $n \in I$ (to minimize E back in section [1] over $(-1,1)$) is a linear combination of the Legendre polynomials in the form $a_0 P_0(x) + a_1 P_1(x) + \dots + a_n P_n(x)$ where each $a_r = \frac{2r+1}{2} \int_{-1}^{1} f(x) \cdot P_r(x) \cdot dx$.[2]

(END SECTION [5])

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D. Richard

SOUTHERN SCHOLARS SENIOR PROJECT

Name: Francis Radnoti Date: 1-18-02 Major: Mathematics

SENIOR PROJECT

A significant scholarly project, involving research, writing, or special performance, appropriate to the major in question, is ordinarily completed the senior year. The project is expected to be of sufficiently high quality to warrant a grade of A and to justify public presentation.

Under the guidance of a faculty advisor, the Senior Project should be an original work, should use primary sources when applicable, should have a table of contents and works cited page, should give convincing evidence to support a strong thesis, and should use the methods and writing style appropriate to the discipline.

The completed project, to be turned in in duplicate, must be approved by the Honors Committee in consultation with the student's supervising professor three weeks prior to graduation. Please include the advisor's name on the title page. The 2-3 hours of credit for this project is done as directed study or in a research class.

Keeping in mind the above senior project description, please describe in as much detail as you can the project you will undertake. You may attach a separate sheet if you wish:

Investigate the application of orthogonal polynomials.

Dr. Richard: project advisor

Signature of faculty advisor [signature] Expected date of completion 4/15/02

Dr. Kevin Brown

Approval to be signed by faculty advisor when completed:

This project has been completed as planned: Yes

This in an "A" project: Yes

This project is worth 2-3 hours of credit: Yes

Advisor's Final Signature Arthur Richert

Chair, Honors Committee ________ Date Approved: ________

Dear Advisor, please write your final evaluation on the project on the reverse side of this page. Comment on the characteristics that make this "A" quality work.